Math 122 Monday, October 17

Finite dimensional V over a field F, $T: V \to V'$ $W = \ker T$
Claim that $\dim V = \dim(\ker T) + \dim(\operatorname{im} T)$ required a lemma

<u>Lemma</u> A subspace $W \subset V$ is <u>finite dimensional</u> $\dim W \leq \dim V$, and if $\dim W = \dim V$ then $W = V$
Pf: Need to show that W has a finite spanning set (of vectors in W). If $W = 0$, done. Else choose $w_1 \neq 0$ in W. If span$\{w_1\} = W$ done. If not, find $w_2 \in W$ linear indep of w_1. If $\{w_1, w_2\}$ span W, done. If not, find w_3. This stops as the size of any linearly independent set of vectors in V is $\leq \dim V$

<u>T-invariant subspaces</u> $T: V \to V$ are subspaces $W \subset V$ such that $T(W) \subset W$. Given a basis $\{e_1, \dots, e_m, e_{m+1}, \dots, e_n\}$ of V where $\{e_1, \dots, e_m\}$ is a basis of W then if W is a T-invariant subspace, its matrix wrt this basis is

$$\begin{array}{c} e_1 \cdots e_m \\ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array}$$

where the upper left hand corner is a $\dim W \times \dim W$ matrix.

Special case: $\dim W = 1$, $W = \{a \cdot v\}$ $a \in F$ $v \neq 0$ in V. Then $Tv = c \cdot v$ where c is an eigenvalue of T and av for $a \neq 0$ are eigenvectors $T(av) = c \cdot a \cdot v$.
Matrix is:
$$\begin{pmatrix} c & \\ 0 & * \\ \vdots & \\ 0 & \end{pmatrix}$$
when v is the first basis element.

Recall T an isomorphism $\Leftrightarrow \ker T = 0 \Leftrightarrow \operatorname{im} T = V \Leftrightarrow \det T \neq 0$ in F
for $T: V \to V$.

Characteristic polynomial of T = poly of deg $n = \dim V$ with coefficients in the field F
$= \det(xI - T) = \det(xI - A)$ wrt any basis of V
$= x^n - Tr(T)x^{n-1} + \cdots + (-1)^n \det T$

<u>Prop</u> c is an eigenvalue for $T \leftrightarrow f(c) = 0$, i.e., c is a <u>root</u> of the char poly $f(x)$.

Pf: c is a root $\Leftrightarrow \det(cI - T) = 0 \Leftrightarrow \ker(cI - T) \neq 0 \Leftrightarrow \exists v \neq 0$ such that $(cI - T)v = 0 \Leftrightarrow Tv = cv$.

<u>Lemma</u> A polynomial of degree n over a field F has at most n roots in F

<u>Not</u> true for a polynomial over rings (add + mult but cannot divide)
ex: $\mathbb{Z}/8\mathbb{Z}$ $f(x) = x^2 - 1 = (x-1)(x+1)$ has four roots $\{1, 3, 5, 7\}$

Pf: If c is a root of $f(x)$ then $f(x)=(x-c)g(x)$ where $g(x)$ is a polynomial of degree $n-1$ over F. This is true by the division algorithm for polynomials. Certainly $f(x)=(x-c)g(x)+r$ for some constant r but then $f(c)=r \Rightarrow r=0$.
Any root of f that's not c must be a root of g, so induct on the degree of f to see that there can be at most n distinct roots.

Aside on polynomial division: $f(x)$ deg n over F, $d(x)$ degree m over F $m \le n$
then $f(x) = d(x)g(x) + r(x)$ where $\deg r < \deg d = m$

Corollary ① Any $T: V \to V$ has at most n distinct eigenvalues.
② If there are n distinct eigenvalues $f(x)=(x-c_1)\cdots(x-c_n)$ then V has a basis of eigenvectors.

Pf of 2) Find $v_1 \neq 0$ in V with eigenvalue c_1, ..., $v_n \neq 0$ in V with eigenvalue c_n.
I claim $\{v_1, \dots, v_n\}$ is a basis for V. Enough to prove that they are linearly independent. Suppose $\sum a_i v_i = 0$ with not all $a_i = 0$! Reorder so that $a_n \neq 0$.
Then $v_n = (-a_n)^{-1}(a_1v_1 + \cdots + a_{n-1}v_{n-1})$. Apply T and write $b_i = (-a_n)^{-1}a_i$. Then
$c_n v_n = c_1 b_1 v_1 + \cdots + c_{n-1}b_{n-1}v_{n-1} = c_n(b_1v_1 + \cdots + b_{n-1}v_{n-1})$. So
$(c_n - c_1)b_1v_1 + \cdots + (c_n - c_{n-1})b_{n-1}v_{n-1} = 0$. Each $c_n - c_i \neq 0$ and so is some b_i
so $\{v_1, \dots, v_{n-1}\}$ is linearly dependent. Induct on n and get a contradiction.

Matrix of T wrt $v_1, \dots, v_n$ is $\begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$ $f(x) = \det \begin{pmatrix} x-c_1 & & & 0 \\ & x-c_2 & & \\ & & \ddots & \\ 0 & & & x-c_n \end{pmatrix} =$

V = sum of n lines W_i each W_i is invariant $(x-c_1)\cdots(x-c_n) = x^n - (\sum c_i)x^{n-1} + (\sum c_ic_j)x^{n-2} \cdots + (-1)^n \prod c_i$.

Hamilton (Cayley) $T: V \to V$ is an element of the
{finite dimensional vector space of all linear maps from $V \to V$} = Hom(V,V)
of dim n^2 = # of independent matrices (note: Hom short for homomorphism)

Hence the set $\{I, T, \dots, T^{n^2}\} \subset \mathrm{Hom}(V,V)$ is linearly dependent so there is $\sum_{i=0}^{n^2} a_i T^i = 0$, i.e., a polynomial of degree n^2 satisfied by T in Hom(V,V)

Cayley-Hamilton Thm T satisfies its own char poly $f(x)$ which has deg n

Hard in general to prove but not too bad for $n=2$.

$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $f(x) = x^2 - (a+d)x + (ad-bc)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)I = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{pmatrix} - \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix} + (ad-bc)I$$

$$= (bc-ad)I + (ad-bc)I = 0$$